Name: ______________________________ #( ) Nov 10, 2022

No notes or other aids are allowed. Read all directions carefully and write your answers in the space provided. **To receive full credit, you must show all of your work. The final answer without steps/work is only worth 0.25 of the points of the question**

1. (7 points) Evaluate the following integral using a trigonometric substitution.

$$\int \frac{1}{x^2\sqrt{x^2+16}}\,dx$$

x^2+4^2

$x = 4\tan\theta \longrightarrow \boxed{x^2 = 4^2\tan^2\theta}$

$x^2+16 = 4^2\tan^2\theta + 4^2 = 4^2(\tan^2\theta+1) = 16\sec^2\theta$

$\boxed{\sqrt{x^2+16} = 4\sec\theta}$ $\boxed{dx = 4\sec^2\theta\, d\theta}$

$$\int \frac{1}{x^2\sqrt{x^2+16}}dx = \int \frac{1}{16\tan^2\theta\, \cancel{4\sec\theta}} \cdot \cancel{4}\sec^{\cancel{2}}\theta\, d\theta$$

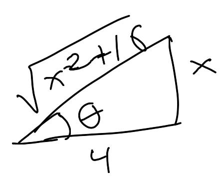
$$= \frac{1}{16}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \frac{1}{16}\int \frac{\cancel{\frac{1}{\cos\theta}}}{\frac{\sin^2\theta}{\cos^{\cancel{2}}\theta}}\,d\theta$$

$$= \frac{1}{16}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$x = 4\tan\theta$

$\tan\theta = \frac{x}{4}$



$u = \sin\theta \longrightarrow du = \cos\theta\, d\theta$

$$= \frac{1}{16}\int \frac{1}{u^2}\,d\theta = \frac{1}{16}\,\frac{-1}{u} + C = \frac{1}{16}\,\frac{-1}{\sin\theta} + C$$

$$= \frac{-1}{16}\cdot\frac{1}{\frac{x}{\sqrt{x^2+16}}}$$

$$= \frac{-\sqrt{x^2+16}}{16x} + C.$$

2. (3 points) Write down the general form of the partial fraction decomposition of the following integral. You are **not required** to solve for the constants in the numerator. **Don't integrate.**

$$\int \frac{x+3}{x^2(x^2+1)^2}\, dx$$

$$\frac{x+3}{x^2(x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

3. (6 points) Evaluate the following integral using partial fraction

$$\int \frac{1}{(x-1)(x^2+1)}\, dx$$

$$\frac{1}{(x-1)(x^2+1)} = \frac{A = 1/2}{x-1} + \frac{Bx+C = -\frac{1}{2}}{x^2+1}$$

$x=1$ $\boxed{A = \frac{1}{2}}$

$x=0$ $\frac{1}{(-1)(1)} = \frac{1/2}{-1} + \frac{C}{1} \Rightarrow -1 = -\frac{1}{2} + C \Rightarrow \boxed{C = -\frac{1}{2}}$

$x=2$ $\frac{1}{(1)(5)} = \frac{1/2}{1} + \frac{2B + -\frac{1}{2}}{5} \Rightarrow \left[\frac{1}{5} = \frac{1}{2} + \frac{2}{5}B + \frac{-1}{10}\right] * 10$

$2 = 5 + 4B - 1$

$-2 = 4B \Rightarrow \boxed{B = -\frac{1}{2}}$

$$\int \frac{1}{(x-1)(x^2+1)}\, dx = \int \frac{1/2}{x-1}\, dx + \int \frac{-\frac{1}{2}x - \frac{1}{2}}{x^2+1}\, dx$$

$$= \frac{1}{2}\int \frac{1}{x-1}\, dx - \frac{1}{2 \cdot 2}\int \frac{2x}{x^2+1}\, dx - \frac{1}{2}\int \frac{1}{x^2+1}\, dx$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\cdot\frac{1}{2}\ln|x^2+1| - \frac{1}{2}\tan^{-1}(x) + C$$

4. (a) (4 points) Approximate the value of

$$\int_1^6 \frac{dx}{x^3}$$

with $n = 5$ subintervals to 6 decimal places using Trapezoidal Rule.

$f(x) = \frac{1}{x^3}$

$\Delta x = \frac{6-1}{5} = 5$

1 2 3 4 5 6

x_1 x_1 x_2 x_3 x_4 x_5

$$T_5 = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right)$$

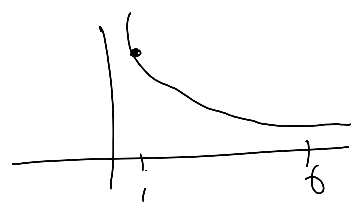
$$= \frac{5}{2}\left(f(1) + 2f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)\right)$$

$$= \frac{5}{2}\left(\frac{1}{1} + \frac{2}{8} + \frac{2}{27} + \frac{2}{64} + \frac{2}{125} + \frac{1}{6^3}\right)$$

(b) (3 points) How large should n be in order to guarantee that the Trapezoidal Rule estimate for $\int_1^6 \frac{dx}{x^3}$ is accurate to within $0.00001 = 10^{-5}$?

$$f(x) = \frac{1}{x^3} = x^{-3} \longrightarrow f'(x) = -3x^{-4} \longrightarrow f''(x) = +12x^{-5}$$

$\boxed{K = 12}$



$$\frac{K(b-a)^3}{12n^2} \leq 10^{-5}$$

$$\frac{12(6-1)^3}{12n^2} \leq 10^{-5} \Rightarrow n^2 \geq \frac{\cancel{12}(6-1)^3}{\cancel{12} \times 10^{-5}} = 12,500,000$$

$$n \geq 3,535.534$$

$$n \geq \boxed{3,536}$$

5. (5 points) Determine if the following integral converges of diverges. Explain why.

$$\int_e^{\infty} \frac{1}{x(\ln(x))^3}\,dx$$

$u = \ln x \longrightarrow du = \frac{1}{x}\,dx$

$$\int \frac{1}{x(\ln x)^3}\,dx = \int \frac{1}{u^3}\,du = \int u^{-3}\,du = \frac{-1}{2}u^{-2} + C$$

$$= \frac{-1}{2(\ln x)^2} + C$$

$$\int_e^{\infty} \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty} \int_e^t \frac{1}{x(\ln x)^3}\,dx = \lim_{t\to\infty} \left.\frac{-1}{2(\ln x)^2}\right|_e^t$$

$$= \lim_{t\to\infty}\left[\left(\frac{-1}{2(\ln t)^2}\right) - \left(\frac{-1}{2(\ln e)^2}\right)\right]$$

$$= \lim_{t\to\infty}\left(\cancel{\frac{-1}{2(\ln t)^2}}^{\,0} + \frac{1}{2}\right)$$

$$= \frac{1}{2}$$

6. (5 points) Compute the arc length of the curve $f(x) = \ln(\sin x)$ over the interval $[\cancel{0}, \cancel{\pi/4}]$.

$a \quad b$

$\frac{\pi}{4}, \frac{\pi}{6}$

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$(\ln g(x))' = \frac{g'(x)}{g(x)}$$

$$f'(x) = \frac{\cos x}{\sin x} = \cot x$$

$$(f'(x))^2 = \cot^2 x \longrightarrow 1 + (f(x))^2 = 1 + \cot^2 x = \csc^2 x$$

$$\sqrt{1 + (f(x))^2} = \sqrt{\csc^2 x} = \csc x$$

$$L = \int_a^b \sqrt{1 + (f(x))^2}\, dx = \int_{\pi/4}^{\pi/6} \csc x \, dx$$

$$= \ln|\csc x - \cot x| \Big|_{\pi/4}^{\pi/6}$$

$$\int \sec x \, dx = \ln|\sec x + \tan x|$$

$$= \ln|\csc \tfrac{\pi}{6} - \cot \tfrac{\pi}{6}| - \ln|\csc \tfrac{\pi}{6} - \cot \tfrac{\pi}{6}|$$

$$\int \csc x \, dx = \ln|\csc x - \cot x|$$

7. Extra credits (**optional**) (4 points) Use the **comparison theorem** to determine whether the improper integral

$$\int_1^\infty \frac{x^2}{\sqrt{x^8+1}}\,dx$$

is convergent or divergent (Show your work!).

see the last page of

section 7.8.